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More rings to rule them all: fragmentation, $4D \leftrightarrow 5D$ and split-spectral flows

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Abstract: In this note we set-up an explicit 5D construction of AdS-fragmentation, whereby a single black ring splits-up into a multi-black ring configuration. Furthermore it is seen that these fragmented rings are equivalent to a direct 5D lift of 4D multi-center black holes. Along the way we also determine the 4D/5D transformations relevant for multi-center charges. It is seen that the physical charges involved in black ring fragmentation are Page charges arising due to 5D Chern-Simons terms. As an application of these methods, we reproduce the total angular momentum of concentric black rings, originally due to Gauntlett and Gutowski. Finally we provide a geometric interpretation of fragmented black rings using the idea of split-spectral flows, which seeks to study charge shifts of a given black ring due to fluxes generated in a multi-ring background.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory.

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Contents

1. Introduction

Starting from (not very) recent work in [4] and [33], a considerable interest has been generated in understanding 5D BPS degeneracies by constructing dualities to the better understood 4D sector [20, 23, 34, 35]. Matter of fact, this 4D/5D relation was put forth by [4] as a 5D version of the OSV conjecture [1]

$$Z^{5D}_{BH} = Z^{4D}_{BH} = |Z_{\text{top}}|^2 \tag{1.1}$$

Evidence for this proposal was sought for by matching the entropy of the 5D BMPV [2] black hole in Taub-NUT space, to the entropy of a 4D Calabi-Yau black hole while making use of the M-theory $\leftrightarrow$ Type II A correspondence. Moreover, since Z^{4D}_{BH} counts degeneracies of single as well as multi-center black holes, it was pointed out by [33] that Z^{5D}_{BH} must also account for equivalent multi-black objects in 5D, assuming eq. (1.1) holds. While a single-center BPS black hole in 4D just lifts to a 5D BMPV black hole; in [33] a rather interesting result was demonstrated: a 4D two-center charge configuration consisting of a D6 charged point-particle at the origin (of $\mathbb{R}^3$) and a D4-D2-D0 charge at a distance $|\vec{L}|$ from it, will in fact lift to a supersymmetric black ring in 5D Taub-NUT space. $|\vec{L}|$ now becomes a modulus on Taub-NUT denoting the distance of the ring from the origin.

On a rather different footing, yet another offshoot of the OSV bandwagon was the work of [3], conceiving baby universes as finite (but still relatively large) N non-perturbative corrections to the OSV conjecture. These corrections go like e^{-N} and are realised as instanton effects in the holographically dual gauge theory. In turn, the holomorphic sector of the gauge theory is dual to the topological string partition sum Z_{top}. The gravitational realisation of these corrections were proposed as 4D multi-center black hole configurations, which can be generated via the mechanism of AdS fragmentation [9, 10] of a single black hole at $x_0 \in \mathbb{R}^3$ into multiple black holes at $\{x_i \in \mathbb{R}^3\}$. These multi-AdS throats are associated to a gravitational instanton action which describes the amplitude for tunneling, in Euclidean time, of a single black hole to multi-black holes. Based on that, [3] forward the idea of a third quantized Hilbert space of baby universes.

One of the motivations driving this note was to reconcile the two aforementioned streams of thought. We try and address some questions regarding the fragmentation of black rings in 5D. Analogous to the 4D case, where we saw how to split D4-D2-D0 charges, here we start with a black ring in Taub-NUT, since this is the pertinent 5D lift of a D4-D2-D0 black hole placed at a distance $|\vec{L}|$ from a single D6 charge (the sole D6 here does not participate in fragmentation). We then set up a fragmentation ansatz for this single ring and see that it splits up into non-concentric multiple black rings (in general). This construction is subject to charge splitting constraints, which as we shall soon see will turn out to be more subtle in the 5D case that they were in 4D due to the presence of cross-terms between multiple centers that must now be carefully tendered.

On the other hand, one might fairly well ask whether the fragmented multi-rings constructed in this manner could as well have been obtained from a direct 5D lift of the 4D multi-center solution. The answer turns out to be in the affirmative; and to do so we shall first reqiure to construct the 4D/5D dictionary for multi-center charges. Compared to the 4D/5D map of [4] for a single black object, the analogous one for multi-centers will turn out a bit more involved again due to the relentless cross-terms. Nevertheless with such a map in hand, transforming amongst 4D/5D multi-center charges, we verify that our fragmented harmonic functions are indeed direct 5D lifts of 4D multi-center solutions. This enables us to confirm commutativity of the following box diagram.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{4D}\\ \text{Black Hole}\end{array}} \arrow[r, "\text{Fragmentation}"] \arrow[d, "4D/5D\ Map"'] & \boxed{\begin{array}{c}\text{4D Multi-}\\ \text{Black Holes}\end{array}} \arrow[d, "4D/5D\ Map"] \\
\boxed{\begin{array}{c}\text{5D}\\ \text{Black Ring}\end{array}} \arrow[r, "\text{Fragmentation}"'] & \boxed{\begin{array}{c}\text{5D Multi-}\\ \text{Black Rings}\end{array}}
\end{tikzcd} \tag{1.2}$$

As had already been hinted by in [3] in context to the 4D set-up; eq. (1.2) seems to predicate the suggestion in 5D, that fragmentation might be thought of as a possible recipe for generating classes of multi-center configurations once given corresponding single-center ones. Of course the multi-rings that we generate in this note by these methods, are by no means any new solutions which had previously been unheard of. For that matter, we point to some of the extensive literature, where several classes of 5D multi-center solutions have been worked out: [19, 24−28]. The focus in this note is based more in the spirit of the box diagram in eq. (1.2) and studying the details therein.

Whilst meandering amidst this impending scheme of things, we are duly confronted with issues concerning the physically meaningful definition of charges in 4D and 5D. We begin with an apprehension of the single black hole/ black ring duality by matching 4D two-center harmonics to 5D black ring harmonics. Such a comparision invokes symplectic charge transformations going from 4D to 5D. Additionally these 4D/5D transformations also make way for an alternative derivation of black ring angular momenta. A clear notion of single-center 4D/5D mapping, now equips us to move on to study the interpretation of 5D multi-center charges. First we procure the 5D charge splitting equations via implementation of the 4D charge splitting equations as well as the single-center 4D/5D lift. The 5D equations so obtained definitely carry the baggage of cross-terms, due to the fact that the 4D/5D transformations are non-linear in the dipole fields. Moreover we shall see that it now becomes relevant to identify which of these charges is of Maxwell type and which of Page type. This discussion picks up from [36] and continues further for the case of fragmented charges. In fact we shall see that in 5D the charges $Q_{Ai(5D)}$ which actually engage in fragmentation are Page charges. These are really the physical multi-ring charges and not the charges $\widetilde{Q}_{A_{i\,(5D)}}$ in terms of which the multi-black ring metric is usually expressed. We also write down an explicit expression transforming between these two types of charges. In due course the multi-center 4D/5D dictionary falls in place.

As an application of charge fragmentation methods described here, we derive the total angular momentum of a system of non-concentric multi-black rings by simply starting from the angular momentum of a single black ring and making use of 5D charge splitting equations. As a check for our answer, we reduce to the special case of concentric black rings in order to compare the our result with the well-known expression of Gauntlett and Gutowski [29, 30]; and yes, their result is correctly reproduced !

The alluring calls for a geometric interpretation of these fragmented rings underscore the final act. In a multi-ring background, individual rings receive multiple spectral flow shifts due to fluxes emanating from split-charge centers; thus coining the notion of 'split-spectral flows'. Each ring may be thought of as sourcing a Dirac string generated due to its magnetic flux. In a Taub-NUT base, these rings are stacked in order of increasing radius. Hence, say the i^{th}-ring; in addition to its own Dirac string; also encircles Dirac strings sourced by each of the $(i-1)$ rings of smaller radius in the Taub-NUT base. And going around Dirac strings is by no means a free ride. It costs large gauge transformations, which can have long-term consequences if Chern-Simons terms are involved as well. This is how spectral flows arise. Therefore the case of our i^{th}-ring multi-timing that many Dirac strings will face a horde of spectral flow shifts to its initial brane charges. This will completely account for the physical split-charges of fragmented rings. Moreover, adding up all the split-spectral flows of all of our wandering fragmented rings correctly gives back the spectral flow of an unfragmented single black ring system, as it should. This sheds light on a geometrical view of the origin of multi-ring Page charges and their cross terms. In fact such split-spectral flows divide the geometry into patches with locally defined gauge field potentials, such that adjacent patches are related upto gauge transformations.

The organisation of this note may not be the most exciting part of this text to read, but still... section 2 provides a lay-out of the 4D multi-center black hole technology and comments on its physical interpretation as baby universes. Section 3 handles harmonics, charges and angular momenta of a single black ring in Taub-NUT from a 4D/5D map. Section 4 is where 5D fragmentation takes shape. We set-up conditions for black ring fragmentation and provide an interpretation for multi-center 5D charges. This follows by writing down a multi-center 4D/5D charge dictionary and also deriving the angular momenta of (non-)concentric multi-black rings. Section 5 seeks to unfold a geometric perspective on the above via the notion of split-spectral flows. Alas, we must wind up...... that's why there's section 6, concluding and throwing pointers at further directions.

2. A glance at 4D black hole fragmentation

In this section we briefly sketch the set-up of 4D black hole fragmentation and its interpretation of baby universes following the approach of [3]. The conceptual basis behind the idea of baby universes lies in the phenomenon of AdS fragmentation [9, 10], which was proposed as an instanton process wherein a single black hole, seen as an excitation in one vacuum configuration, tunnels to a multi-black hole state appearing as an excitation in another vacuum. The two vacua lie in the asymptotic limits of a "Euclidean time" co-ordinate, which is defined by an entropy functional $S(x)$. From the Euclidean metric (obtained after a Wick rotation) the $AdS_2 \times S^2$ geometry is seen to flow to a product geometry of $\otimes_{i=1}^{n} AdS_2^i \times S_i^2$ (to leading approximation).

As an explicit representation of multi-black hole configurations, the authors of [3] make use of the well-known multi-center solutions of $\mathcal{N}=2$ supergravity from [5−8]. The idea behind the fragmentation procedure is that the black hole harmonic functions interpolate between the single-center harmonics; at asymptotic infinity $x \to \infty$; and the multi-center harmonics; which are achieved upon approaching the near-horizon limit. In fact, near the i^{th}-horizon when $x \to x_i$, the i^{th}-black hole dominates the solution. Therefore given a single-center solution and implementing the above idea, one can set up an ansatz for harmonic functions of fragmented black holes. Additionally charge conservation constrains the distribution of charges at fragmented centers. In [3] it was shown that the supergravity configuration of [5−8] can indeed be realised in this way via AdS fragmentation of a single black hole. For the sake of setting up notation as well as later reference, let us flash a quick glance at how this works.

Consider the harmonic functions of a single black hole in 4D with magnetic charges p^I and electric charges q_I placed at the spatial origin in $\mathbb{R}^3$

$$U^I(x) = \frac{p^I}{|x|} + u^I \qquad\qquad V_I(x) = \frac{q_I}{|x|} + v_I \tag{2.1}$$

here $I = 0, 1, \ldots\ldots$ denotes vector multiplet indices; $x \in \mathbb{R}^3$; and u^I, v_I are constants determined at infinity. In these co-ordinates the pole at $x = 0$ is the location of the horizon which has the topology of a two-sphere S^2. Another ingredient we will require is the entropy functional $S(x) \equiv S\left[U^I(x), V_I(x)\right]$. This is a specific polynomial function of the harmonics and only at the horizon does it attain the value of the entropy. Elsewhere $S(x)$ freely flows between its asymptotic limits. This flow in $S(x)$ will induce the harmonic functions $U^I(x)$, $V_I(x)$ to interpolate between single-center and multi-center solutions. At asymptotic infinity with $x \to \infty$, a single black hole geometry with charges p^I, q_I placed at the origin and harmonics given by eq. (2.1) leads to $S(x) \to c$ (a finite number). When these harmonics are inserted in the metric we get the well-known topology of $AdS_2 \times S^2$ and $S(x)$ enters this near-horizon Bertotti-Robinson metric as the square of the AdS_2 radius. The idea of AdS fragmentation now proposes treating $S(x)$ as a Euclidean time direction. Then the $S(x) \to \infty$ asymptote serves as another vacuum into which there exists a finite probability amplitude for a single black hole system to tunnel into a system of multi-black holes. The most general solution for harmonic functions, which interpolate between these asymptotic vacua, looks like

$$U^I(x) = \sum_{i=1}^{n} \frac{p_i^I}{|x - x_i|} + u^I \qquad\qquad V_I(x) = \sum_{i=1}^{n} \frac{q_{Ii}}{|x - x_i|} + v_I \tag{2.2}$$

where $U^I(x)$, $V_I(x)$ now describe a multi-black hole system with n horizons located at centers $\{x_i\}$. Charge splitting is subject to the following constraints

$$\sum_{i=1}^{n} p_i^I = p^I \qquad \text{and} \qquad \sum_{i=1}^{n} q_{Ii} = q_I \tag{2.3}$$

To fully specify the solution additional integrability conditions are also required

$$\left(p_i^I V_I(x) - q_{Ii} U^I(x)\right)\Big|_{x=x_i} = 0 \tag{2.4}$$

which have to be evaluated at each horizon. Now we can see how the above harmonic functions interpolate between single and multi-center geometries as follows: at asymptotic infinity $x \to \infty$, the harmonics in eq. (2.2) reduce to eq. (2.1) (by using the constraints in eq. (2.3)) and $S(x) \to c$; whereas at each $x \to x_i$, only the i^{th}-summands in eq. (2.2) dominate, describing multiple black holes located at $\{x_i\}$ respectively and consequently giving $S(x) \to \infty$. Hence flowing $S(x)$ from c to ∞ describes an AdS geometry fragmenting into a multi-AdS geometry. Eqs. (2.2), (2.3) and (2.4) were originally derived as part of the multi-center $\mathcal{N} = 2$ supergravity solution of [5]. In [3] this solution has been interpreted as remnants of an AdS fragmentation process.

3. Black ring from 4D/5D duality

In this section we demonstrate how charges, harmonics and angular momenta of a black ring can be determined purely in terms of a 4D/5D duality. While the charges and harmonics are straightforward to get; an explicit expression for ring angular momenta obtained from 4D/5D lifting will serve to compliment the usual supergravity derivations discussed in the literature.

We start by considering the following two-center system in 4D:

$$
\begin{aligned}
U^0(x) &= \frac{1}{|x|} + u^0 & \qquad V_A(x) &= \frac{q_{A(4D)}}{|x-x_0|} + v_A \\
U^A(x) &= \frac{p^A_{(4D)}}{|x-x_0|} + u^A & \qquad V_0(x) &= \frac{q_{0(4D)}}{|x-x_0|} + v_0
\end{aligned} \tag{3.1}
$$

which consists of a single D6 charge ($p^0_{(4D)} = 1$) at the origin $x = 0$ (in $\mathbb{R}^3$); and $p^A_{(4D)}$, $q_{A(4D)}$ and $q_{0(4D)}$ respectively D4, D2, D0 charges, which form a 4D black hole at $x = x_0$. In [33], the 4D metric of this system is decompactified to yield a 5D black ring in Taub-NUT. Instead of doing that, here we go for a more direct comparison; namely, showing that the 4D harmonics above will be identical to 5D black ring harmonics once they are expressed via 5D charges. For this we will require the 4D/5D charge transformations

$$p^A_{(4D)} = p^A_{(5D)} \tag{3.2}$$

$$q_{A(4D)} = \left(Q_{A(5D)} - 3D_{ABC}p^B_{(5D)}p^C_{(5D)}\right) \tag{3.3}$$

where $Q_{A(5D)}$ and $p^A_{(5D)}$ respectively will turn out to be black ring electric and magnetic charges. We shall soon comment on their interpretation. An additional ingredient required to specify eq. (3.1) are the integrability conditions, which yield

$$v_0 = -\frac{q_{0(4D)}}{L} \tag{3.4}$$

$$q_{0(4D)} = v_A p^A_{(4D)} \left(\frac{1}{L} + \frac{4}{R^2_{TN}}\right)^{-1} \tag{3.5}$$

here L denotes the radial distance $|x_0|$. The presence of a D6-brane leads to a geometric transition when lifting to M-theory, giving a Taub-NUT space in the uncompactified directions. Therefore $U^0(x)$ becomes a harmonic function in Taub-NUT with $u^0 = \frac{4}{R^2_{TN}}$ (with R_{TN} as the asymptotic radius of Taub-NUT). u^A remains arbitrary and can be set to zero. Putting all this together, the 4D harmonics above can indeed be compared to the known Taub-NUT-black ring harmonics in the literature [20][1] (see also [16])

$$
\begin{aligned}
H_{TN}(x) &= \frac{4}{R^2_{TN}} + \frac{1}{|x|} & \qquad L_A(x) &= v_A + \frac{Q_{A(5D)} - 3D_{ABC}{p^B}_{(5D)}{p^C}_{(5D)}}{|x-x_0|} \\
K^A(x) &= \frac{{p^A}_{(5D)}}{|x-x_0|} & \qquad M(x) &= \frac{J_{\rm tube}}{L} + \frac{-J_{\rm tube}}{|x-x_0|}
\end{aligned} \tag{3.6}
$$

where $J_{\rm tube} \equiv -q_{0(4D)}$, which is determined from eq. (3.5), is indeed the intrinsic (not total) angular momentum of the ring along the S^1 circle and is the M-theory lift of the D0-charge. Thus the harmonic functions of the 4D two-center system under consideration are exactly equivalent to those of a 5D black ring in Taub-NUT[2].

[1] Compared to [20] we have scaled the p^A_{5D} charge by a factor of (-1).

[2] A black ring in $\mathbb{R}^4$ (see [14, 15, 17, 18]) can be extracted as a special case of eq. (3.6) by taking the limit $R_{TN} \to \infty$. The conventions of [14, 15] differ from [20] by rescaling of charges; in this note we continue using the latter.

Note that these functions in eq. (3.6) (along with integrability conditions) completely specify the black ring solution. For the sake of completeness, let us quickly demonstrate how this comes about. Consider the most general 5D $\mathcal{N} = 1$ ungauged supergravity solution [31, 32] which is given by the following 5D metric and gauge fields

$$\begin{aligned} ds_5^2 &= -f^2(dt+\omega)^2 + f^{-1}ds^2(M_4) \\ F^A &= d\left[fX^A(dt+\omega)\right] - \frac{2}{3}fX^A(d\omega + \star d\omega) \end{aligned} \tag{3.7}$$

where X^A are scalar fields in abelian vector multiplets; they satisfy the constraint equation $D_{ABC}X^AX^BX^C = 1$ and X_A are defined by the condition $X^AX_A = 1$. $ds^2(M_4)$ in the equation above refers to the Gibbons-Hawking metric of a 4D hyper-Kahler base space, which in our case is simply taken to be $ds^2(TN)$, the Taub-NUT metric (or $ds^2(\mathbb{R}^4)$ when considering a black ring in flat space). Let r, θ, ϕ, ψ denote coordinates on the 4D base space with (r,θ,ϕ) locally parameterising an $\mathbb{R}^3$ and ψ running along a compact S^1 with periodicity 4π. The Hodge dual $\star$ is taken with respect to the 4D base space. The function f and one-form ω are fully nailed down in terms of four yet-to-be-specified harmonic functions as follows

$$\begin{aligned} f^{-1}X_A &= \frac{1}{4}{H_{TN}}^{-1}D_{ABC}K^BK^C + L_A \\ \omega &= \left(-\frac{1}{8}{H_{TN}}^{-2}D_{ABC}K^AK^BK^C - \frac{3}{4}{H_{TN}}^{-1}K^AL_A + M\right) \\ &\quad \times(d\psi + cos\theta d\phi) + \widehat{\omega} \end{aligned} \tag{3.8}$$

The notation used in this equation is intentionally suggestive. Furthermore, $\widehat{\omega}$ is defined by

$$\nabla \times \widehat{\omega} = H_{TN}\nabla M - M\nabla H_{TN} + \frac{3}{4}\left(L_A\nabla K^A - K^A\nabla L_A\right) \tag{3.9}$$

Now inserting the explicit form of the harmonic functions of eq. (3.6) into eqs. (3.7), (3.8) and (3.9) simply reproduces the complete black ring solution of [20] in Taub-NUT (or [16] in $\mathbb{R}^4$). Moreover, operating the gradient on both sides of eq. (3.9) and evaluating at the poles, exactly recovers the integrability conditions of eq. (2.4), which are subsequently solved to get eqs. (3.4) and (3.5). This prescription goes through for multi-rings as well. Inserting appropriate multi-ring harmonics into the same 5D supergravity metric given above, one can recover the multi-black ring solution [29, 30]. In this sense, the harmonics and integrability conditions can be said to be sufficiently representative of the solutions of single as well as multi-black rings. For what follows here, we shall adopt this stance as well. Therefore the focus in this note shall not be on solving supergravity equations themselves, but rather on obtaining quantities such as multi-ring harmonics, charges and angular momenta from ring fragmentation and spectral flows.

Now coming back to the 4D/5D transformations, a comment on eqs. (3.2) and (3.3) is due. These equations were derived in [4] by considering symplectic shifts in electric charges due the presence of a magnetic flux such that the degeneracy of microstates remains invariant. Subsequently this leads to matching of leading order entropies for 4D and

JHEP02(2008)066

5D black holes. Also, the authors of [20 − 22] further clarify these transformations when interpolating from a 4D black hole to a 5D black ring. While $q_{A(4D)}$ is the observable in 4D, from the 5D perspective it is $Q_{A(5D)}$ which is the observed charge. Let us point out to yet another interpretation of these transformations coming from spectral flow shifts (as in [12]) associated to the 5D Chern-Simons term. In a later section, we pursue this last observation further.

Much like the above-mentioned D2 charges, there also occurs a shift for D0 charges (again due to [4])

$$q_{0(5D)} = q_{0(4D)} - (p^A_{(4D)} q_{A(4D)} + D_{ABC} p^A_{(4D)} p^B_{(4D)} p^C_{(4D)}) \tag{3.10}$$

Starting from this relation we now obtain an independent identification of the total black ring angular momenta. Simply using eqs. (3.2), (3.3), (3.4) and (3.5) into eq. (3.10) yields

$$q_{0(5D)} = v_A p^A_{(5D)} \left(\frac{1}{L} + \frac{4}{R^2_{TN}} \right)^{-1} - p^A_{(5D)} (Q_{A(5D)} - 2D_{ABC} p^B_{(5D)} p^C_{(5D)}) \tag{3.11}$$

Now let us denote $q_{0(5D)} \equiv -\frac{G}{3\pi} J_\psi$, where G is the 5D Newton's constant. Then J_ψ exactly compares to the total angular momentum of the ring along the S^1 circle as given in [20] (or [14] on reducing to $\mathbb{R}^4$). The first term of J_ψ is the intrinsic angular momentum arising via the presence of D0 charges along the S^1 circle (ψ-direction); the second component describes the angular momentum induced in the presence of a magnetic flux. In addition there is yet another angular momentum characterising the ring; one associated to the ϕ-circle along the S^2, perpendicular to the ψ-circle. In the absence of D0 charges along the ϕ-circle, with only flux going through it, the angular momentum contribution (denote as J_ϕ) is solely flux-induced, thus giving

$$J_\phi = J_\psi - \frac{3\pi}{G} J_{\text{tube}} \tag{3.12}$$

Thus far we conclude that explicit application of the 4D/5D correspondence correctly identifies the charge prescription, harmonic functions as well as angular momenta of a black ring. Proceeding this way the leading order black ring entropy too can be obtained, as well as its one-loop correction. Since the references [21, 22, 33, 34] do justice to the former and [23] to the latter, we shall have no more to say on that. Equipped with these tools, we shall next test their application for the case of multi-center black holes/rings.

4. Black ring fragmentation and 5D charge splitting

As seen in section 2, 4D charge fragmentation is given by simple linear relations in terms of fragmented charges. For D4, D2, D0 branes respectively, we denote these splittings as

$$\sum_{i=1}^{n} p_i^A{}_{(4D)} = p^A{}_{(4D)} \qquad \sum_{i=1}^{n} q_{Ai(4D)} = q_{A(4D)} \qquad \sum_{i=1}^{n} q_{0i(4D)} = q_{0(4D)} \tag{4.1}$$

Let us note that in 4D these are also the physically observed charges. We would now like to construct the analog of these equations in 5D. In that case, as we shall soon see, the charge fragmentation equations are not only non-linear (in the dipole charges) but also involve cross-term contributions arising from multiple charge centers.

4.1 5D multi-black ring charges and harmonic functions from fragmentation

Owing to the trivial 4D/5D relation for magnetic charges ${p^A}_{(5D)}$ (as in eq. (3.2)), their splitting into 5D components is straightforward.

$$
{p^A}_{(5D)} = \sum_{i=1}^{n} {p_i^A}_{(5D)} \tag{4.2}
$$

The more interesting case is that of the electric charge $Q_{A(5D)}$ of a single black ring. Since this charge differs from the corresponding 4D charge $q_{A(4D)}$ by large gauge transformations induced via the Chern-Simons term in the 5D action, therefore the 5D splitting for $Q_{A(5D)}$ will turn out to be more involved. Analogous to the 4D case, let us define the fragmentation of this charge to be

$$
Q_{A(5D)} \equiv \sum_{i=1}^{n} Q_{Ai(5D)} \tag{4.3}
$$

where we now have to determine $Q_{Ai(5D)}$ and then provide it with a physical interpretation. To do this, we substitute the conditions given in eq. (4.1) into eqs. (3.2) and (3.3). Upon further rearranging we get

$$
\begin{aligned}
Q_{A(5D)} &= \left\{ \sum_{i=1}^{n} q_{Ai(4D)} + \sum_{i=1}^{n}\sum_{j=1}^{n} 3D_{ABC}{p_i^B}_{(4D)}{p_j^C}_{(4D)} \right\} && (4.4)\\
&= \sum_{i=1}^{n} \left\{ \left(\widetilde{Q}_{A_i\,(5D)} - 3D_{ABC}{p_i^B}_{(5D)}{p_i^C}_{(5D)} \right) + \sum_{j=1}^{n} 3D_{ABC}{p_i^B}_{(5D)}{p_j^C}_{(5D)} \right\} && (4.5)
\end{aligned}
$$

where the last line has been converted to 5D quantities with the intent of extracting 5D charge fragments. $\widetilde{Q}_{A_i\,(5D)}$ is introduced as a new 5D variable defined by the following 4D/5D transformation

$$
\widetilde{Q}_{A_i\,(5D)} = q_{Ai(4D)} + 3D_{ABC}{p_i^B}_{(4D)}{p_i^C}_{(4D)} \tag{4.6}
$$

Notice that the right-hand side of eq. (4.5) has been expressed in a way that facilitates comparison to the literature. $\widetilde{Q}_{A_i\,(5D)}$ is actually a 5D charge associated to the i^{th} black ring and is the one that appears in the usual 5D multi-ring supergravity solutions (for instance see [29, 30]). In this way, eq. (4.5) is simply the ADM mass[3] of [29, 30]. Note that because these references dwell in conventions different from ours, the following rescaling of charges must be used: ${p_i^A}_{(5D)} \longrightarrow \sqrt{2}{p_i^A}_{(5D)}$. Also they use C_{ABC} as the intersection number, which relates to the one used here via $C_{ABC} = 6D_{ABC}$.

Despite the above comparison, let us remark that in our case eq. (4.5) is obtained as a result of 5D fragmentation. Therefore it is clear that summing all the $\widetilde{Q}_{A_i\,(5D)}$'s over all

[3] Even though [29, 30] only refer to concentric rings, the above comparison is still meaningful because effects due to non-concentricity only start showing up for quantities involving the position vector $\vec{L}$, such as angular momentum, entropy, etc.

i would not conserve $Q_{A(5D)}$. The charges that are actually involved in 5D fragmentation are clearly the $Q_{Ai(5D)}$'s and not $\widetilde{Q}_{A_i\,(5D)}$'s. So the question arises, which of these two is the correct physical observable ? In order to answer this, we shall take a closer look at the interpretation of each of these charges via their integral representations. It will turn out that it is in fact the $Q_{Ai(5D)}$'s that are the physically observable quantities and not the $\widetilde{Q}_{A_i\,(5D)}$'s. The subtlety between $Q_{Ai(5D)}$ and $\widetilde{Q}_{A_i\,(5D)}$ arises precisely due to the presence of cross-terms relating different charge centers. The consequences of these cross-terms will also be evident in other quantities such as multi-ring angular momenta. For later reference, let us note down the relation between the two charges

$$Q_{Ai(5D)} = \widetilde{Q}_{A_i\,(5D)} + \sum_{j=1}^{i-1} 3D_{ABC}\left(p^B_{i\;(5D)}p^C_{j\;(5D)} + p^B_{j\;(5D)}p^C_{i\;(5D)}\right) \tag{4.7}$$

Whilst plucking this expression from eq. (4.5) one has also to keep in mind that $Q_{Ai(5D)}$ should be independent of how the cycles B and C have been labelled. Therefore the resulting expression for $Q_{Ai(5D)}$ has to be symmetrised as done above.

Now let us try to understand the various 5D charges discussed above in the form of integrals over near-horizon patches. In [36] it was shown that in terms of purely near-horizon fields (and not requiring data from the complete solution) of a single black ring, the charge $Q_{A(5D)}$ can be understood as a Page charge rather than a Maxwell charge (see also [37] for a clear exposition on the different notions of charge)

$$Q_{A(5D)} = \int_\Sigma \left(\star a_{AB}F^B + 3D_{ABC}A^B\wedge F^C\right) \tag{4.8}$$

where the range of integration, denoted by Σ, is a spatial 3-cycle in the vicinity of the black ring horizon. The a_{AB}, which is a function of the scalar moduli, serve the usual purpose of lowering vector multiplet indicies. A^B denote near-horizon U(1) gauge fields around the black ring. A Page charge is conserved, localised and quantised, but not gauge invariant. The near-horizon integral on the right-hand side of eq. (4.8) implicitly represents a Page charge. In [36], they explicitly compute this integral and show that it indeed results in the black ring charge $Q_{A(5D)}$.

Adapting the results of [36] to the present context of fragmented rings, we now argue that the $Q_{Ai(5D)}$'s are also Page charges. This is consistent with the role of eq. (4.3) as a charge conservation equation. Then $Q_{Ai(5D)}$ should also have an expression as a localised charge resulting from a near-horizon integral

$$Q_{Ai(5D)} \stackrel{?}{=} \int_{\Sigma_i} \left(\star a_{AB}F^B_i + 3D_{ABC}A^B_i\wedge F^C_i\right) \tag{4.9}$$

for A^B_i as U(1) gauge fields locally defined in the neighbourhood of the i^{th}-ring horizon. Σ_i denotes a 3-cycle enclosing the i^{th}-horizon and $F^B_i = dA^B_i$. So the question then is: does this integral in eq. (4.9) work out to give $Q_{Ai(5D)}$? Upon inserting the following expression for the gauge field:

$$A_i^B = -\left[\left(D^{BC}\left(\widetilde{Q}_{A_{i\,(5D)}} - 3D_{CDE}p_i^D{}_{(5D)}p_i^E{}_{(5D)}\right) + 2\sum_{j=1}^{n} p_j^B{}_{(5D)}\right)d\psi \right.$$
$$\left. + \left(p_i^B{}_{(5D)}(1+x) + 2\sum_{j=i+1}^{n} p_j^B{}_{(5D)}\right)d\chi\right] \tag{4.10}$$

into the integral in eq. (4.9), the authors of [36] indeed do obtain the expression[4] we had in eq. (4.7). In eq. (4.10), the variables ψ, χ and x are the usual ring-coordinates (notation follows from [14]). $(\psi+\chi/2)$ and x parametrise the S^2, while $(\psi-\chi/2)$ runs along the S^1 near the horizon of the i^{th}-black ring. The gauge fields A_i^B are locally defined patchwise. Gluing of adjacent patches is achieved via gauge transformations. In eq. (4.10), $i = 1$ refers to the innermost ring (smallest radius) and the radial parameter monotonically increases with increasing i. The expression for A_i^B used in [36] was extracted from the supergravity solution of [29, 30] for concentric black rings. The same is reliable for non-concentric rings too, since restrictions to concentricity mainly become relevant when evaluating integrability conditions (and those bear consequences for multi-ring angular momenta).

From the expression for A_i^B above, we see that the gauge field around the i^{th}-ring also feels the back-reaction due to dipole fields from neighbouring rings. It is precisely this dipole field back-reaction that leads to cross-terms in the computation of $Q_{Ai(5D)}$. In our case we tried to derive these terms from the construction of 5D fragmentation. It is gratifying to note that they exactly compare with those coming from the integral of [36]. As we shall see that fragmentation of a single ring indeed does reproduce the correct multi-ring charges.

Now turning our attention to $\widetilde{Q}_{A_{i\,(5D)}}$, let us see why this is in fact not a physical charge. From the definition of $\widetilde{Q}_{A_{i\,(5D)}}$ in eq. (4.6), its 4D/5D transformation is identical to that of a single black ring system with electric charge $\widetilde{Q}_{A_{i\,(5D)}}$ and magnetic charge $p_i^A{}_{(5D)}$. This is in stark contrast to the analogous transformation of $Q_{Ai(5D)}$ (which can be read-off from eq. (4.4)). Unlike $Q_{Ai(5D)}$, we see that $\widetilde{Q}_{A_{i\,(5D)}}$ clearly does not sense the background reaction due to neighbouring rings. Hence such a charge cannot be given a global physical meaning in a multi-ring geometry. Its presence is at best only a local approximation. Therefore its integral representation is trivially identical to eq. (4.8) after all charges (which enter into the explicit expressions for the gauge potentials) have been replaced by those at the i^{th}-center.

Eqs. (4.4) and (4.6) essentially describe the multi-center 4D/5D dictionary for electric charges. As expected the physical multi-center Page charge $Q_{Ai(5D)}$ transforms in a more complicated way than $Q_{A(5D)}$ (eq. (3.3)), due to the multi-black ring background. On the other hand, the charges $\widetilde{Q}_{A_{i\,(5D)}}$, though unphysical, retain manifest symplectic invariance of the original single-center solution. Each of the $\left(p_i^A{}_{(5D)}\,,\,\widetilde{Q}_{A_{i\,(5D)}}\right)$ manifestly transform

[4]In [36] the computation was first done for the special case of only one vector field, and then it was generalised to n U(1) fields by simply carrying through the same calculation with vector indicies.

as a symplectic pair. This underlying property often makes it convenient to express multi-black ring solutions in terms of these charges (as has been usual practice in the literature).

Having explicitly constructed the 5D charge fragmentation equations for magnetic and electric charges $\left(p^A{}_{(5D)}\,,\, Q_{A(5D)} \right)$ along with the relevant multi-center 4D/5D transformations, we are now equipped to derive two of the multi-ring harmonic functions $\left(K^A(x), L_A(x)\right)_{multi}$ from the single-ring harmonics $\left(K^A(x), L_A(x)\right)_{\text{single}}$ by merely implementing the fragmentation recipe of section 2. As in eqs. (2.2) and (2.3) we have

$$\begin{aligned} L_A(x)\Big|_{\text{single}} &= v_A + \frac{Q_{A(5D)} - 3D_{ABC}p^B{}_{(5D)}p^C{}_{(5D)}}{|x-x_0|} \\ &\longrightarrow v_A + \sum_{i=1}^{n} \frac{Q^{\bullet}_{Ai(5D)} - 3D_{ABC}p^{\bullet B}_{\ i\ (5D)}p^{\bullet C}_{\ i\ (5D)}}{|x-x_i|} = L_A(x)\Big|_{multi} \end{aligned} \tag{4.11}$$

which is subject to the constraint

$$Q_{A(5D)} - 3D_{ABC}p^B{}_{(5D)}p^C{}_{(5D)} = \sum_{i=1}^{n} \left(Q^{\bullet}_{Ai(5D)} - 3D_{ABC}p^{\bullet B}_{\ i\ (5D)}p^{\bullet C}_{\ i\ (5D)} \right) \tag{4.12}$$

Eqs. (4.11) and (4.12) constitute a natural 5D fragmentation ansatz with newly-defined charges $Q^{\bullet}_{Ai(5D)}$ and $p^{\bullet A}_{\ i\ (5D)}$ such that at $x \to \infty$ one recovers $L_A(x)\Big|_{\text{single}}$ while at $x \to x_i$ the solution (at leading approximation) appears like a single black ring at the i^{th} location. Now the constraint in eq. (4.12) above is identical in form to the charge splitting eq. (4.5), which suggests the identification

$$Q^{\bullet}_{Ai(5D)} \equiv \widetilde{Q}_{A_i\,(5D)} \qquad p^{\bullet A}_{\ i\ (5D)} \equiv p_i^A{}_{(5D)} \tag{4.13}$$

From this we also see how the charges $\widetilde{Q}_{A_i\,(5D)}$ enter into the 5D harmonics and subsequently into the metric. Of course the above harmonic function could also have been written in terms of $Q_{Ai(5D)}$, but then the expressions would only get a little messy as we proceed.

Another remark that we can make at this stage is that eq. (4.11) (along with the conditions in eqs. (4.12) and (4.13)) could also have been obtained via a different route; namely, by direct use of the multi-ring 4D/5D transformation (eq. (4.6)) into eqs. (2.2) and (2.3). This is consistent with the commutativity of the diagram in eq. (1.2), which suggests that fragmenting a single black ring into multiple black rings reproduces the same configuration as that obtained by a direct 5D lift of the appropriate 4D multi-center black holes.

Dealing with the harmonic function $K^A(x)$ for magnetic charges is now straightforward:

$$\begin{aligned} K^A(x)\Big|_{\text{single}} &= \frac{p^A{}_{(5D)}}{|x-x_0|} \\ &\longrightarrow \sum_{i=1}^{n} \frac{p_i^A{}_{(5D)}}{|x-x_i|} = K^A(x)\Big|_{\text{multi}} \end{aligned} \tag{4.14}$$

which is again subject to

$$p^A{}_{(5D)} = \sum_{i=1}^{n} p_i^A{}_{(5D)} \tag{4.15}$$

As per the other two black ring harmonic functions $H_{TN}(x)$ and $M(x)$: the former remains unchanged under fragmentation as our brane configuration includes only a single D6 charge (which lifts to a Kaluza-Klein monopole in 5D); while fragmentation of the latter comes up in the following sub-section.

4.2 Multi-black ring angular momenta from ring fragmentation

We are now ready to derive the expressions for angular momenta of a multi-black ring system from 5D fragmentation techniques. Our starting point is eq. (3.11): the angular momentum of a single black ring along the ψ-direction

$$J_\psi = \frac{3\pi}{G} J_{\rm tube} + \frac{3\pi}{G} p^A_{(5D)} (Q_{A(5D)} - 2D_{ABC} p^B_{(5D)} p^C_{(5D)}) \tag{4.16}$$

Inserting the 5D charge splitting eqs. (4.2) and (4.5) into the above we readily obtain

$$\begin{aligned} J_\psi \; = \; \frac{3\pi}{G} \sum_{i=1}^{n} J^i_{\rm tube} + \frac{3\pi}{G} \Bigg[& \sum_{i,j=1}^{n} p_i^A{}_{(5D)} \left(\widetilde{Q}_{A_j\,(5D)} - 3D_{ABC} p_j^B{}_{(5D)} p_j^C{}_{(5D)} \right) \\ & + \sum_{i,j,k=1}^{n} D_{ABC} p_i^A{}_{(5D)} p_j^B{}_{(5D)} p_k^C{}_{(5D)} \Bigg] \end{aligned} \tag{4.17}$$

where the quantities $J^i_{\rm tube}$ have yet to be determined from integrability conditions. As a special case of our result in eq. (4.17), we shall be able to reproduce the expression for angular momentum of concentric black rings which was first derived by Gauntlett and Gutowski in [29, 30] in the context of 5D supergravity. In order to obtain $J^i_{\rm tube}$, we will first have to determine the multi-ring harmonic function $M(x)$, from where $J^i_{\rm tube}$ can be extracted. Therefore, fragmenting the function $M(x)$ yields

$$\begin{aligned} M(x)\Big|_{\rm single} \;&=\; v_0 + \frac{-J_{\rm tube}}{|x - x_0|} \\ &\longrightarrow v_0 + \sum_{i=1}^{n} \frac{-J^i_{\rm tube}}{|x - x_i|} = M(x)\Big|_{\rm multi} \end{aligned} \tag{4.18}$$

subject to

$$J_{\rm tube} = \sum_{i=1}^{n} J^i_{\rm tube} \tag{4.19}$$

Additionally, the multi-ring harmonics $(H_{TN}(x), K^A(x), L_A(x), M(x))_{\rm multi}$ above also have to satisfy integrability conditions as in eq. (2.4). These are to be evaluated at each horizon. Starting with $x = 0$, we get

$$v_0 = \sum_{i=1}^{n} \frac{J^i_{\rm tube}}{L_i} \tag{4.20}$$

This determines the constant v_0 in terms of $J^i_{\rm tube}$ (which we still have to fix in terms of electric and magnetic charges) and L_i (which is the radial distance in $\mathbb{R}^3$ of the $i^{\rm th}$ pole from the origin). However, as discussed earlier, v_0 is a constant predetermined at infinity and should not be affected by the process of fragmentation. As a consistency check we shall see in what follows that eq. (4.20) is indeed identical to eq. (3.4) obtained earlier in section 3. Before that we will require to compute the remaining n conditions at the horizons $\{x_i\}$. This yields

$$\begin{aligned}-J^i_{\rm tube} = \left(\frac{4}{R^2_{TN}} + \frac{1}{L_i}\right)^{-1} \Bigg(& p_i^A{}_{(5D)} v_A + \sum_{j=1, j\neq i}^{n} \frac{p_i^A{}_{(5D)}\left(\widetilde{Q}_{A_j\,(5D)} - 3D_{ABC}p_j^B{}_{(5D)}p_j^C{}_{(5D)}\right)}{\sqrt{L_i^2 - 2L_iL_j cos\theta_{ij} + L_j^2}} \\ & - \sum_{j=1, j\neq i}^{n} \frac{\left(\widetilde{Q}_{A_i\,(5D)} - 3D_{ABC}p_i^B{}_{(5D)}p_i^C{}_{(5D)}\right)p_j^A{}_{(5D)}}{\sqrt{L_i^2 - 2L_iL_j cos\theta_{ij} + L_j^2}}\Bigg) \end{aligned} \tag{4.21}$$

where θ_{ij} is the angle between $\vec{L}_i$, $\vec{L}_j \in \mathbb{R}^3$. Now rearranging eq. (4.21) for $\frac{J^i_{\rm tube}}{L_i}$ and then inserting back into eq. (4.20) produces

$$v_0 = -\frac{4\,J_{\rm tube}}{R^2_{TN}} - v_A p^A{}_{(5D)} \tag{4.22}$$

after also using eqs. (4.2) and (4.19). Indeed eq. (4.22) is precisely the value of v_0 obtained earlier by inserting eq. (3.5) into eq. (3.4).

Now with eqs. (4.21) and (4.22) the function $M(x)\Big|_{\rm multi}$ is fully specified. Thus simply from 5D black ring fragmentation we were able to construct all of the multi-black ring harmonic functions. Moreover inserting eq. (4.21) for $J^i_{\rm tube}$ into eq. (4.17) results in the complete expression for the total multi-black ring angular momentum in the ψ-direction: J_ψ. Also the angular momentum in the ϕ-direction: J_ϕ, can then be read-off from J_ψ since

$$J_\phi = J_\psi - \frac{3\pi}{G}\sum_{i=1}^{n} J^i_{\rm tube} \tag{4.23}$$

still continues to hold.

An additional comment on eq. (4.21) is due. Let us take a closer look at the last two terms on the right-hand side of this equation. As long as the multi-center charges are constrained to remain mutually non-local, then $\vec{L}_i \neq \vec{L}_j$ will hold and that avoids any potential singularity in eq. (4.21). Hence the sum of the two numerators (within the summation symbols) is allowed to assume any non-zero value. From the 4D point of view, this is precisely the condition for the dual 4D charges $(p_i^A{}_{(4D)}, q_{Ai(4D)})$ to be non-parallel (on the charge lattice). This was the interesting new feature in the multi-center solution of [5 – 8]. On the other hand, if the condition $\vec{L}_i \neq \vec{L}_j$ were to be relaxed; then we would be required to impose

$$p_i^A{}_{(5D)}\left(\widetilde{Q}_{A_j\,(5D)} - 3D_{ABC}p_j^B{}_{(5D)}p_j^C{}_{(5D)}\right) - \left(\widetilde{Q}_{A_i\,(5D)} - 3D_{ABC}p_i^B{}_{(5D)}p_i^C{}_{(5D)}\right)p_j^A{}_{(5D)} = 0 \tag{4.24}$$

for all $i \neq j$, thereby eliminating the last two terms in eq. (4.21). The corresponding 4D charge vectors $(p^A_{i\,(4D)}, q_{Ai(4D)})$ are now parallel-aligned on the charge lattice.[5] The reason we made the above comment is because the construction in [29, 30] does restrict to eq. (4.24) and hence we too will need to make use of it whenever comparing to their results. For all other purposes, our results continue to hold for non-parallel charges in general.

Eq. (4.17) along with eq. (4.21) gave us the most general result for the angular momentum (along the ψ-coordinate) of non-concentric multi-black rings. We would now like to reduce our result to the case of concentric rings so as to compare it with the well-known answer of [29, 30], which was derived using 5D supergravity techniques of [31] and [32]. First we set all angles θ_{ij} between the poles to zero. The co-linear alignment of poles in $\mathbb{R}^3$ translates to concentric rings in 5D. In order to eliminate Dirac-Misner strings, [29, 30] choose to impose eq. (4.24), which can be interpreted as a restriction to parallel 4D charge vectors.[6] From our discussion above, we see that it is still possible to continue with non-parallel charges by trading-off mutual locality of charges. Nevertheless to make contact with [29, 30]; we use eq. (4.24) in eq. (4.21) with all angles $\theta_{ij} = 0$ and thus arrive at the desired result upon plugging everything back into eq. (4.17). To facilitate a direct comparison, let us also connect with the notation of [29, 30]; which is achieved via simple charge redefinitions. Firstly we note that the 4D/5D transformations - eqs. (3.3) and (3.10) - match their counterparts in [29, 30] after the following two redefinitions: $q_{0(5D)} \longrightarrow (q_{0(5D)} + p^A_{(4D)} q_{A(4D)})/2$ and $p^A_{(4D)} \longrightarrow \sqrt{2} p^A_{(4D)}$. We have already seen how the latter conformed to 5D split-charges and played a role in matching eq. (4.5) to the above literature. Now coming to the multi-ring angular momentum in eq. (4.17), it can be seen after some algebra that the first of the above two redefinitions simply gives a factor of 2 to the last term of eq. (4.17). Then making use of the second redefinition in the form $p^A_{i\,(5D)} \longrightarrow \sqrt{2} p^A_{i\,(5D)}$ produces

$$\begin{aligned} J_\psi = -\frac{6\sqrt{2}\pi}{G} \sum_{i=1}^{n} L_i p^A_{i\,(5D)} v_A + \frac{\sqrt{2}\pi}{G} \Bigg[& 3 \sum_{i,j=1}^{n} p^A_{i\,(5D)} \left(\widetilde{Q}_{A_{j\,(5D)}} - C_{ABC} p^B_{j\,(5D)} p^C_{j\,(5D)} \right) \\ & + 2 \sum_{i,j,k=1}^{n} C_{ABC} p^A_{i\,(5D)} p^B_{j\,(5D)} p^C_{k\,(5D)} \Bigg] \end{aligned} \tag{4.25}$$

which exactly agrees (upto an overall factor which we leave to one's taste) with [29, 30] as the total angular momentum of concentric black rings in $\mathbb{R}^4$.

[5] Note that being parallel on the charge lattice should not be confused with co-linearity of the poles in $\mathbb{R}^3$. Even for parallel charges the multi-center poles are still free to remain non-colinear. From a 4D/5D perspective, non-colinear D4-D2-D0 poles in 4D lift to non-concentric rings in 5D.

[6] In fact this is not the most general way to eliminate Dirac-Misner strings and admittedly ends up making the solution of [29, 30] highly restrictive. In general it suffices to impose the integrability conditions as we have done in this note. The difference with [29, 30] is that those authors impose eq. (4.21) in a very special way.

Finally let us remark that writing the 5D charge $q_{0(5D)}$ in the form

$$q_{0(5D)} = \sum_{i=1}^{n} q_{0i(5D)} \tag{4.26}$$

its fragments can be easily read-off from eq. (4.17) above. Just as was the case earlier with the $Q_{Ai(5D)}$ charge, we see again that the multi-ring 4D/5D transformations for $q_{0i(5D)}$ are more complicated due to the presence of cross-terms which must be carefully taken into account while performing a 4D/5D lift. In the next section, we proceed to discuss the physical origin of these cross-terms and their geometric interpretation.

5. Geometric interpretation using split-spectral flows

In this section we try to provide a geometric understanding of multi-black rings, based on successive application of spectral flow transformations. Such split-spectral flows now assume relevance in the presence of multiple $AdS_3 \times S^2$ horizons. This generalises the spectral flow discussions of [12, 13] to a multi-center setting.

Let us first consider a single black ring, whose near-horizon geometry is $AdS_3 \times S^2$. This will be seen to fit exactly within the considerations of [12, 13]. In this background geometry, the 5D supergravity action contains a Chern-Simons term

$$S_{CS} = \int_{AdS_3 \times S^2} D_{ABC} A^A \wedge F^B \wedge F^C \tag{5.1}$$

which is not invariant under large gauge transformations. $F^A = dA^A$ is the usual two-form U(1) magnetic flux passing through the S^2. The electric charge is obtained by varying the 5D action with respect to the field strength F^A. Due to the presence of the above-mentioned Chern-Simons term, the electric charge so obtained also varies under large gauge transformations

$$q_A = \int_{S^2 \times S^1} \Big(\star F_A + 3 D_{ABC} A^B \wedge F^C \Big) \tag{5.2}$$

Since the 5D supergravity action can be obtained from a Calabi-Yau compactification of M-theory, the electric charge q_A is the M2-brane charge from a 11-dimensional perspective (or D2 charge in Type II A) and the magnetic charge p^A defined as

$$p^A = \int_{S^2} F^A \tag{5.3}$$

is the M5-brane charge[7] (or D4 in 10 dimensions).

It can be seen by inspection that the second term in the integrand in eq. (5.2) will decay rapidly when evaluated over a homologous 3-surface sufficiently distant from the

[7]Strictly speaking, this definition remains valid so long as the NUT charge (the KK monopole at the origin) is not encompassed by the S^2.

horizon, leaving only the first term to contribute. However, prior to integration, let us consider the effect of a large gauge transformation of A^A, of the type

$$A^A \longrightarrow A^A + k^A d\,(\psi/2\pi) \tag{5.4}$$

with k^A an integer and $0 \leq \psi \leq 2\pi$ a coordinate running along the S^1. This leaves us with A^A-independent terms that do not vanish at infinity, thereby producing shifts in the electric charge q_A of the type

$$q_A \longrightarrow q_A + 3D_{ABC}k^B p^C \tag{5.5}$$

This charge is clearly not gauge invariant and the physical explanation shall soon follow. For now, let us note that this equation compares to the 4D/5D charge transformation that we encountered earlier in eq. (3.3), since it is the lack of gauge invariance of the 5D Chern-Simons term in the action that is responsible for inducing shifts in the original gauge invariant 4D charges.

Similarly the M-theory angular momentum q_0 (or D0 charge in Type II A) is again not a gauge invariant quantity and we now proceed to derive its charge shifts, obtained via gauge transforming an integral representation of angular momentum. For a 5D supergravity action (to be thought of as a semi-classical reduction of M-theory in our context), such an integral can be extracted from appropriate contributions to the gauge field energy-momentum tensor. For the aforesaid 5D action, this has been derived in [36] making use of Wald's method [38]

$$q_0 = -\int_{S^2 \times S^1} \left(\star\, d\xi + \star(\xi \cdot A^A)\, F_A + D_{ABC}\,(\xi \cdot A^A)\, A^B \wedge F^C \right) \tag{5.6}$$

Here ξ denotes the axial Killing vector with respect to the ψ-direction, while $(\xi \cdot A^A)$ is an interior product between a vector field and a one-form. The Killing field ξ generates isometries along the ψ-direction; leading to a conserved charge, which is the angular momentum. In fact, the right-hand side of eq. (5.6) is simply the Noether charge of Wald. Asymptotically, the A^A dependent terms in the integrand (in eq. (5.6)) drop off and the integral reduces to Komar's formula for the angular momentum. However, large gauge transforming with eq. (5.4) yields precisely two more asymptotically non-vanishing remnants. Recognizing the asymptotic form of eq. (5.2) and eq. (5.3) leads to the following charge shifts in angular momentum

$$q_0 \longrightarrow q_0 - k^A q_A - D_{ABC}k^A k^B p^C \tag{5.7}$$

This again can be compared to the 4D/5D transformation in eq. (3.10).

Now eqs. (5.5), (5.7) are in fact the spectral flow transformations in question. The name spectral flow arises due to the fact that in the dual $(0, 4)$ SCFT these transformations correspond to automorphisms of the conformal algebra. Moreover spectral flow is a symmetry of the theory as it leaves the generalised elliptic genus of the CFT invariant. Note that such flows are characteristic of an odd dimensional theory. For a 4D black hole with $AdS_2 \times S^2$ horizon, the supergravity action is gauge invariant. Therefore the

JHEP02(2008)066

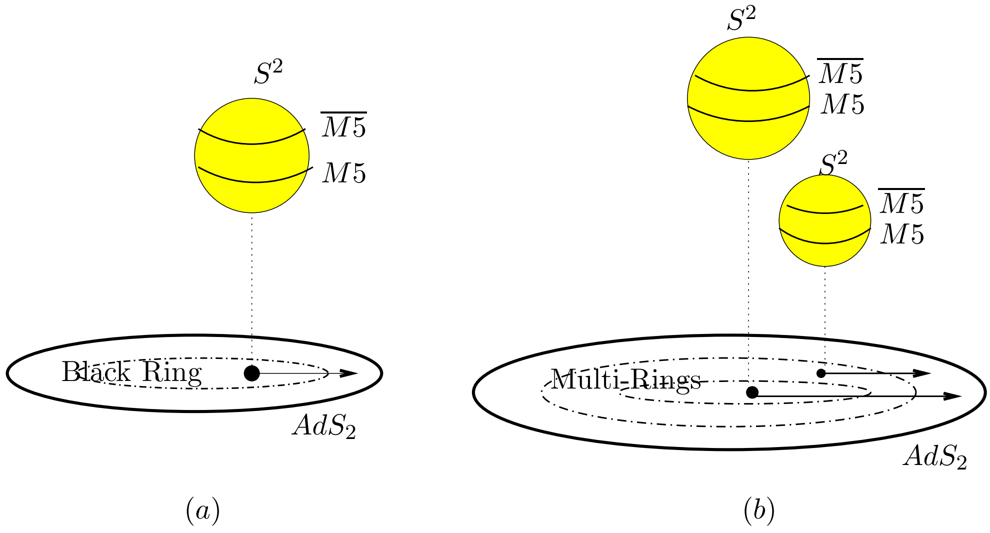

Figure 1: Visualising the spectral flow for black rings: (a) Nucleation of an M5-anti-M5 pair around a single black ring leading to a large gauge transformation. (b) The same idea now extended to a multi-black ring background leads to multiple gauge transformations in a geometrically ordered way.

electric charge equals the actual number of D2 branes wrapped on Calabi-Yau 2-cycles; while the D0 charge counts the physical D0 branes. Because of this we can also interpret eqs. (5.5), (5.7) as a 5D lift of 4D charges.

The gauge transformation in eq. (5.4) is picked up upon going around (perpendicular to the ψ-direction) the ring with a probe particle; which has been given the interpretation of M5-anti-M5 branes being pair-produced, going around the ring in opposite directions and mutually annihilating (see figure 1 in [12]). More precisely, this can be visualised as follows. The spatial near-horizon geometry of a bound state of M5-M2 branes (with angular momentum) is a product of Euclidean AdS_2 and S^2 (refer to figure 5.1 (a) below). On the AdS_2 disc, the black ring is depicted as a circle along the ψ-direction. The radial coordinate on the disc is the same as the Taub-NUT radial direction. Now consider the pair-production of k^A M5-anti-M5 pairs. These wrap 4-cycles on the Calabi-Yau, while the fifth direction goes around the equator of an S^2. This S^2 is a point on the AdS_2 disc, located on the inside of the circle representing the black ring. The M5-anti-M5 rings along the S^2 equator move apart in opposite directions towards the poles, where they self-annihilate leaving behind a Dirac surface on the S^2. Since the location of the Dirac surface is unphysical, it can be moved away to spatial infinity. Upon crossing the ring, it causes a shift of gauge potential by an amount $k^A d\left(\psi/2\pi\right)$. Thus the presence of a magnetic flux k^A shifts the guage potential A^A and consequently the charges q_A, q_0. For the case of the single ring described above, this flux is the dipole flux passing through the ring and is generated by its own M5 charges. Hence $k^A = p^A$ here, which leads to eqs. (5.5), (5.7).

JHEP02(2008)066

5.1 Electric charges and split-spectral flows

Now extending the above discussion, we shall systematically derive multi-black ring electric charges and angular momenta as a split-spectral flow argument. We begin with electric charges. Let us label the n rings with an index i, in increasing order of radius. The innermost ring is labeled $i = 1$. Its brane charges are p_1^A, q_{A_1}, q_{0_1}. Here p_1^A exhibits a dipole behaviour, generating a magnetic flux $k^A = p_1^A$. This in turn shifts q_{A_1} by spectral flow as in eq. (5.5). Indeed this innermost ring behaves just like the single ring case encountered in the previous discussion above. Moving onto the next ring, this has brane charges p_2^A, q_{A_2}, q_{0_2}. As depicted in figure 5.2 below, the total flux passing through this ring is not only that generated by its own charge p_2^A, but also that emanating from the inner ring. These distinct fluxes give rise to the following spectral flows:

$$\begin{array}{ccc} & & \delta = 2, \gamma = 2 \\ & \nearrow & \text{with} k^B = p_2^B \\ q_{A_2} \longrightarrow q_{A_2} + 3D_{ABC} p_\delta^B p_\gamma^C & \longrightarrow & \delta = 1, \gamma = 2 \\ & & \text{with} k^B = p_1^B \\ & \searrow & \delta = 2, \gamma = 1 \\ & & \text{with} k^C = p_1^C \end{array} \tag{5.8}$$

where the last transformation occurs due to the fact that the flux has also to be symmetrised with respect to the cycles. The physical electric charge of this ring is then obtained by adding up all these shifts to the original brane charge.

From the point of view of figure 5.1 (b), multi-rings are depicted as n-circles on the disc, one inside the other. Nucleation of an M5-anti-M5 pair now occurs in the vicinity of each of the n rings, creating n Dirac surfaces. Upon moving these surfaces to infinity, the i^{th}-ring is crossed by i Dirac surfaces each with flux p_j^A, giving in total a flux $k_{tot}^A = \sum_{j=1}^{i} p_j^A$ passing through this ring. This is the origin of multiple spectral flows for a multi-ring system.

We can now directly write down the result for the i^{th}-ring with all the spectral flows put together: those resulting from the intrinsic (due to ring's own magnetic charge) flux as well as those from background (generated by those rings which are encircled by the i^{th} one) flux, we get

$$q_{A_i} \longrightarrow q_{A_i} + 3D_{ABC} p_i^B p_i^C + 3D_{ABC} \sum_{j=1}^{i-1} (p_i^B p_j^C + p_j^B p_i^C) \tag{5.9}$$

Much like the analogy in electrostatics, the fluxes due to rings which encircle the i^{th}-ring from the outside, do not affect it. With respect to figure 5.1 (b), each ring acts as a source, emanating flux; while the sink is at infinity. Hence only those rings placed to the exterior of the source ring will lie in its flux field. Eq. (5.9) gives us the physical charge of the

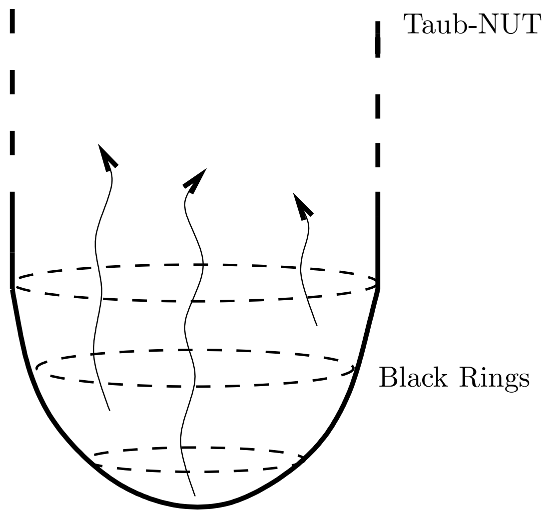

Figure 2: A Taub-NUT perspective of the influence of magnetic flux generated by individual black rings upon neighbouring black rings.

$i^{\rm th}$-ring from a spectral flow analysis. This can be compared to eq. (4.7), where the same quantity emerged from a fragmentation analysis.

Furthermore upon adding up the split-spectral flow shifts of all of the n rings leads to the total spectral flow shift of the full multi-ring configuration

$$\begin{aligned} Q_A^{total} &\equiv \sum_{i=1}^{n} q_{A_i} + 3D_{ABC}\sum_{i=1}^{n} p_i^B p_i^C + 3D_{ABC}\sum_{i=1}^{n}\sum_{j=1}^{i-1}(p_i^B p_j^C + p_j^B p_i^C) \\ &= \sum_{i=1}^{n} q_{A_i} + 3D_{ABC}\sum_{i,j=1}^{n} p_i^B p_j^C \end{aligned} \tag{5.10}$$

where in the last step, the identity

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1}(A_{ij}+A_{ji}) \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} - \sum_{i=1}^{n} A_{ii} \tag{5.11}$$

has been used. Indeed Q_A^{total} exactly equates to $Q_{A(5D)}$ in eq. (4.4), which is simply the electric charge of a single black-ring system. Therefore, adding up all the spectral flow shifts as well as the total brane charge gets us back to the geometry of a single black ring. In this sense the spectral flow transforms of a multi-ring system are really split-spectral flows of a single ring system.

5.2 Angular momenta and split-spectral flows

The multi-ring angular momentum can now be obtained in a similar fashion. Once again consider the $i^{\rm th}$-ring with brane charges p_i^A, q_{A_i}, q_{0_i}. The relevant angular momentum spectral flows for this ring are

$$q_{0_i} \longrightarrow q_{0_i} - p_i^A \sum_{j=1}^{i} q_{A_j} - D_{ABC} p_i^A p_i^B p_i^C \tag{5.12}$$

$$q_{0_i} \longrightarrow q_{0_i} - \sum_{j=1}^{i-1} p_j^A q_{A_i} - D_{ABC} \sum_{j=1}^{i-1} p_j^{(A} \sum_{k=1}^{i-1} p_k^B p_i^{C)} \tag{5.13}$$

In the above flow equations, firstly we have the intrinsic magnetic flux $k_i^A = p_i^A$, generated by M5 charges on the i^{th}-ring itself. This flux interacts with M2 charges as well as M5 charges (carried on other Calabi-Yau cycles), both on the i^{th}-ring. Then there is the background magnetic flux $k_{back}^A = \sum_{j=1}^{i-1} p_j^A$ because this ring is placed in the background fields generated by the $i-1$ rings to its interior. Now a new addition to the above is a background electric flux $\sum_{j=1}^{i-1} q_{A_j}$, which also interacts with electric charges on the i^{th}-ring. That explains the second term on the right-hand side of eq. (5.12). And eq. (5.13) then accounts for interactions of the magnetic background with the i^{th}-brane charges in the usual way. The last term there has to be symmetrised and therefore the brackets in superscripts denote a sum over all symmetric permutations of cycles. Then adding up all these contributions will result in the angular momentum of the i^{th}-ring.

To get the total angular momentum of the multi-ring system we add up those of each of the rings

$$\begin{aligned}
J^{total} \equiv & \sum_{i=1}^{n} q_{0_i} - \sum_{i=1}^{n} \sum_{j=1}^{i-1} (p_i^A q_{A_j} + p_j^A q_{A_i}) - \sum_{i=1}^{n} p_i^A q_{A_i} - D_{ABC} \sum_{i=1}^{n} p_i^A p_i^B p_i^C \\
& - D_{ABC} \sum_{i=1}^{n} \sum_{j=1}^{i-1} p_j^{(A} \sum_{k=1}^{i-1} p_k^B p_i^{C)} \\
= & \sum_{i=1}^{n} q_{0_i} - \sum_{i,j=1}^{n} p_i^A q_{A_j} - D_{ABC} \sum_{i,j,k=1}^{n} p_i^A p_j^B p_k^C
\end{aligned} \tag{5.14}$$

Upon substituting q_{A_j} in the last equality above with $\widetilde{Q}_{A_{i\,(5D)}}$ via eq. (4.6), we see that eq. (5.14) indeed compares[8] to eq. (4.17) leading to $J^{total} = -\frac{G}{3\pi} J_\psi$. A split-spectral flow analysis thus provides us with a physical understanding of where all the different multi-ring angular momentum contributions actually come from. In particular, it gives a clear description of how individual rings behave in the background of other rings.

Consequently a geometric picture of this multi-black ring configuration emerges from such split-spectral flow considerations. In fact what these split-flows are really doing is to break up the global multi-ring geometry into patches with locally defined gauge potentials; such that gauge fields in neighbouring patches are related upto large gauge transformations. In figure 5.1 (b) these patches can be identified as follows: first there's the innermost disc inside the first ring, defining a patch with gauge potential A_1^A; then there are the annular

[8] Of course spectral flow does not determine q_{0_i} as a function of L_i. That input still relies on the integrability conditions.

regions all around it, with gauge potentials A_2^A, A_3^A,... respectively. This defines a chain of potentials spanning the entire geometry

$$A_1 \xrightarrow{\beta_1} A_2 \xrightarrow{\beta_2} A_3 \cdots\cdots\cdots\cdots \xrightarrow{\beta_{n-1}} A_n \xrightarrow{\beta_n} A_n + \beta_n \tag{5.15}$$

(suppressed vector indices may be readily reinstated here) the β_i are large gauge transformations between A_i and A_{i+1}. In fact these local regions emerging here due to split-spectral flow considerations might provide a conceptual basis for the analysis of [36] where the authors compute localised charge integrals for black rings by dividing the geometry into local patches which are all glued together. The existence of such patches enable near-horizon integrals such as those in eqs. (4.8), (4.9) to capture all the data normally extracted from the full geometry.

6. Conclusions and outlook

Two remarkable set of ideas pertaining to string theoretic descriptions of black holes, that have generated lots of excitement in the aftermath of the OSV conjecture are: (1) the 4D/5D connection between black holes/rings [4, 33]; and (2) multi-center black holes as non-perturbative corrections to the black hole partition function [3]. In this note we have sought for a modest attempt at combining these two, in the sense of the commutative box diagram of eq. (1.2).

We approach the problem by setting-up an explicit 5D construction of black ring fragmentation and thereafter also show that fragmented black rings are equivalent to a direct 5D lift of 4D multi-black holes. For the purposes of the latter, we determine the multi-center 4D/5D charge transformations as well.

Related to these events is the important issue of interpretation of charges in 5D, especially for our multi-center split charges. In [36] it was shown that the electric charge (and angular momentum) of a single black ring could be expressed purely in terms of near-horizon data as a Page charge. In our analysis we see that the 5D charges $Q_{Ai(5D)}$ which participate in fragmentation are in fact also Page charges (as opposed to being Maxwell charges) and in that sense these are the physical charges of the system. Whereas the multi-center charges $\widetilde{Q}_{A_{i\,(5D)}}$ that usually appear in the supergravity multi-ring metric are not physical charges. Even though the latter-mentioned charges can be algebraically related to the former ones, we find it nevertheless important to distinguish the physically relevant ones for the multi-ring configuration.

A rather interesting application of the 5D fragmentation methods developed in this note is an alternative derivation of the angular momenta of concentric black rings. It is indeed gratifying to note that we are able to exactly reproduce the results of Gauntlett and Gutowski.

Lastly, we saw how the introduction of split-spectral flows lends a geometric perspective to shifts in brane charges of fragmented black rings by accounting how a Dirac string generated by a given ring influences other rings in such a multi-ring background. This serves as yet another derivation for the total angular momentum of a multi-ring system. Moreover summing up all the split-spectral flow shifted charges of all the fragmented rings

JHEP02(2008)066

exactly gives back the observed electric charge of a single black ring. The split-spectral flows basically divide the geometry into patches with locally defined gauge fields. The significance of these patches becomes relevant when computing near-horizon integrals.

From a broader perspective, one might contemplate over the role of fragmented configurations on the black hole/ring partition function. In [3], each fragmented configuration is viewed as a multi-AdS throat geometry; and further following [9, 10], each such geometry is associated to some saddle point of the partition function. In that sense Z_{BH} is presumed to sum over all possible geometries subject to charge conservation constraints. Fragmentation is thus a euclidean tunneling process from one minima to another. These leading order contributions therefore dominate the multi-AdS partition sum of [3]. However there ought to be further sub-leading corrections to each multi-center configuration that should be computable from any complete partition sum. At this stage, it would be very tempting to think that the black hole farey tail partition function of [11 − 13] might be precisely the object that captures the multi-center saddle points as well as its sub-leading corrections. Whether or not these multi-center geometries lend a physical description to the farey tail story remains to be seen.

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